

DIMENSIONAL REDUCTION AND THE EQUIVARIANT CHERN CHARACTER

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ABSTRACT. We propose a dimensional reduction procedure in the Stolz–Teichner framework of supersymmetric Euclidean field theories (EFTs) that is well-suited in the presence of a finite gauge group or, more generally, for field theories over an orbifold. As an illustration, we give a geometric interpretation of the Chern character for manifolds with an action by a finite group.

1. INTRODUCTION

In the context of topological quantum field theory (in the functorial sense of Atiyah and Segal), dimensional reduction is the assignment of a $(d-1)$ -dimensional theory to a d -dimensional theory induced by the functor of bordism categories $S^1 \times -: (d-1)\text{-Bord} \rightarrow d\text{-Bord}$. In the Stolz–Teichner framework of supersymmetric Euclidean field theories (EFTs) [17, 18], dimensional reduction is a more subtle subject, but it can still be made sense of and provides geometric interpretations of classical constructions in algebraic topology. To give the basic idea, we first recall that $0|1$ -dimensional EFTs over a manifold X are in bijection, after passing to concordance classes, with de Rham cohomology classes of X [12]. On the other hand, super parallel transport [8] allows us to associate a field theory $E_V \in 1|1\text{-EFT}(X)$ to any vector bundle with connection $V \in \text{Vect}^\nabla(X)$, and a similar statement relating $1|1$ -dimensional EFTs and topological K -theory is widely expected. Moreover, there is a dimensional reduction map red between (groupoids of) field theories over X that recovers the Chern character, in the sense that the diagram

$$\begin{array}{ccccc} & & E & \rightarrow & 1|1\text{-EFT}(X) & \xrightarrow{\text{red}} & 0|1\text{-EFT}(X) \\ & \nearrow & & & \downarrow & & \downarrow \\ \text{Vect}^\nabla(X) & & & & K^0(X) & \xrightarrow{\text{ch}} & H^{\text{ev}}(X; \mathbb{C}) \end{array}$$

commutes [11, 7].

This paper is part of an ongoing project aiming to identify gauged supersymmetric field theories as geometric cocycles for equivariant cohomology theories [16, 4, 5]. Our main goal here is to extend the above dimensional reduction procedure for $1|1$ -EFTs to the case where the manifold X is replaced by an orbifold \mathfrak{X} (or, more generally, any stack on the site SM of supermanifolds). This will be based on a series of functors between variants of the Euclidean bordism categories over \mathfrak{X} ,

$$(1) \quad 0|1\text{-EBord}(\Lambda\mathfrak{X}) \leftarrow 0|1\text{-EBord}^\mathbb{T}(\Lambda\mathfrak{X}) \rightarrow 0|1\text{-EBord}^{\mathbb{R}/\mathbb{Z}}(\Lambda\mathfrak{X}) \rightarrow 1|1\text{-EBord}(\mathfrak{X}).$$

The two middle objects, which we call \mathbb{T} - respectively \mathbb{R}/\mathbb{Z} -equivariant bordisms over the inertia stack $\Lambda\mathfrak{X}$, as well as the maps involving them, are introduced in section 3. Here, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ stands for the circle group and \mathbb{R}/\mathbb{Z} is the stack arising

from the action of \mathbb{Z} on \mathbb{R} . These are of course equivalent as group stacks, and our terminology just intends to indicate which model for the circle is directly involved in the definition of each bordism category. (The two equivariant bordism categories also turn out to be equivalent, though this requires proof; see theorem 4.) Dimensional reduction of field theories (or twist functors) will then be realized as the pull-push operation induced by diagram (1).

As a simple but hopefully illustrative application, we specialize to the case where $\mathfrak{X} = X//G$ is a global quotient orbifold and give a field-theoretic interpretation of the simplest instance of orbifold Chern character, namely the one concerning untwisted cohomology of global quotients [1]. It is easy to extend the map E above for an orbifold \mathfrak{X} in place of the manifold X . (For the sake of brevity, we will only describe the partition function Z_V of the field theory E_V , cf. section 4.2.) From the discussion of section 2 it will follow that 0|1-EFTs over the inertia orbifold $\Lambda\mathfrak{X}$ are geometric cocycles for the so-called delocalized cohomology $H_G^{\text{ev}}(\hat{X})$ —the codomain of the equivariant Chern character (cf. section 4.1). Finally, in section 4.3 we verify that the dimensional reduction of E_V corresponds to a form representing $\text{ch}_G(V)$.

Theorem 1. *Let $\mathfrak{X} = X//G$ be the quotient stack arising from the action of a finite group on a manifold. Then the diagram*

$$\begin{array}{ccccc} & & E \rightarrow & 1|1\text{-EFT}(\mathfrak{X}) & \xrightarrow{\text{red}} & 0|1\text{-EFT}(\Lambda\mathfrak{X}) \\ & \nearrow & & & & \downarrow \\ \text{Vect}^\nabla(\mathfrak{X}) & & & & & \\ & \searrow & & K_G^0(X) & \xrightarrow{\text{ch}_G} & H_G^{\text{ev}}(\hat{X}; \mathbb{C}) \end{array}$$

commutes.

In a subsequent paper, we will construct twists for 1|1-EFTs over \mathfrak{X} associated to classes in $H^3(\mathfrak{X}, \mathbb{Z})$, using a representing gerbe with connection as input data, and relate the dimensional reduction of these twists, as well as the corresponding twisted field theories, with more general versions of the orbifold Chern character (see [15] for an earlier version of this story).

While this work was in preparation, closely related preprints by Daniel Berwick-Evans appeared on the arXiv [4, 3]. His approach is heavily inspired by ideas from perturbative quantum field theory, while ours is more geometric, putting group actions on stacks at the forefront.

1.1. Terminology and background. For an extensive survey of the Stolz–Teichner program, see [17]. The facts more directly relevant to this paper, regarding 0|1-dimensional field theories, are in [12]. Concerning supermanifolds, we generally follow the definitions and conventions of Deligne and Morgan [6], and in particular we routinely use the functor of points formalism. The necessary facts about Euclidean structures are reviewed in appendix B.

We treat stacks on the site SM of supermanifolds (where a covering is a collection of jointly surjective local diffeomorphisms) in a geometric way, meaning, for instance, that most of our diagrams involving manifolds must be interpreted as diagrams in stacks, where some of the objects happen to be representable sheaves. We recommend the appendix of Hohnhold et al. [12] for a short introduction to stacks, and Behrend and Xu [2] as a more detailed reference, including the stacky perspective on orbifolds and cohomology of orbifolds. The less standard piece of descent theory needed in this paper concerns group actions on stacks. We offer a short overview

(with further references) in appendix A, where we also record a lemma that may be of independent interest (proposition 8).

Vector bundles are always $\mathbb{Z}/2$ -graded and over \mathbb{C} , and C^∞ , Ω^* denote the sheaves of complex-valued functions and differential forms on SM.

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2. BORDISMS AND FIELD THEORIES OVER AN ORBIFOLD

A d -dimensional topological (quantum) field theory, in the usual definition of Atiyah and Segal, is a symmetric monoidal functor

$$E \in \text{Fun}^\otimes(d\text{-Bord}, \text{Vect})$$

between the category of d -dimensional bordisms and the category of vector spaces. The domain has as objects closed $(d-1)$ -dimensional manifolds and as morphisms diffeomorphism classes of bordisms between them.

Stolz and Teichner [17] consider a refinement of the above, where each bordism is equipped with several additional geometric structures: supersymmetry, meaning that a bordism is now a supermanifold of dimension $d|\delta$; a Euclidean structure in the sense of appendix B; and finally a smooth map to a fixed manifold X . In order to make sense of the idea that field theories should depend smoothly on the input data, we are led to formulate the resulting bordism category $d|\delta\text{-EBord}(X)$ as a (weak) category internal to symmetric monoidal stacks. This also allows us to keep track of isometries between bordisms instead of just considering equivalence classes of bordisms modulo isometry.

Once this framework is in place, it is clear how to replace the manifold X by a “generalized manifold”, or stack, \mathfrak{X} : an S -family of bordisms in $d|\delta\text{-EBord}(\mathfrak{X})$ is given by a submersion $\Sigma \rightarrow S$ of codimension $d|\delta$ with fiberwise Euclidean structure, an object of \mathfrak{X}_Σ (which, by the Yoneda lemma, corresponds to a map $\psi: \Sigma \rightarrow \mathfrak{X}$ in the realm of generalized manifolds), and lastly some boundary information we will not detail here. A morphism over $f: S' \rightarrow S$ in the stack of bordisms is determined by a fiberwise isometry $F: \Sigma' \rightarrow \Sigma$ covering f (and suitably compatible with the boundary information) together with a morphism ξ between objects of $\mathfrak{X}_{\Sigma'}$ as indicated in the diagram below.

$$\begin{array}{ccccc} \Sigma' & & \xrightarrow{\psi'} & & \mathfrak{X} \\ & \searrow F & \downarrow \xi & \searrow \psi & \\ & & \Sigma & & \\ \downarrow & & & & \downarrow \\ S' & \xrightarrow{f} & S & & \end{array}$$

Finally, Euclidean field theories of dimension $d|\delta$ over \mathfrak{X} are functors of internal categories:

$$d|\delta\text{-EFT}(\mathfrak{X}) = \text{Fun}^\otimes(d|\delta\text{-EBord}(\mathfrak{X}), \text{TV}),$$

where TV is an internal version of the category of topological vector spaces. These are contravariant objects on the variable \mathfrak{X} , and we call two EFTs $E_0, E_1 \in d|\delta\text{-EFT}(\mathfrak{X})$

concordant if there exist a field theory $E \in d|\delta\text{-EFT}(\mathfrak{X} \times \mathbb{R})$ such that $E \cong \text{pr}_1^* E_0$ on $\mathfrak{X} \times (-\infty, 0)$ and $E \cong \text{pr}_1^* E_1$ on $\mathfrak{X} \times (1, \infty)$.

These observations are the foundation of an equivariant extension of Stolz–Teichner program. In this paper, we are only interested in the cases $d|\delta = 0|1$ or $1|1$, so we can work with simplified definitions, which we discuss in the remainder of this section.

2.1. Dimension $0|1$. Since every $0|1$ -dimensional bordism is closed, we can define

$$0|1\text{-EFT}(\mathfrak{X}) = \text{Fun}_{\text{SM}}(\mathfrak{B}(\mathfrak{X}), \mathbb{C}) = C^\infty(\mathfrak{B}(\mathfrak{X})),$$

where $\mathfrak{B}(\mathfrak{X})$ is a model for the full substack of comprising fiberwise connected bordisms in $0|1\text{-EBord}(\mathfrak{X})$, namely

$$\mathfrak{B}(\mathfrak{X}) = \Pi T\mathfrak{X} // \text{Isom}(\mathbb{R}^{0|1}), \text{ where } \Pi T\mathfrak{X} = \underline{\text{Fun}}_{\text{SM}}(\mathbb{R}^{0|1}, \mathfrak{X}).$$

Here, Fun_{SM} denotes the groupoid of fibered functors and natural transformations over SM, while $\underline{\text{Fun}}_{\text{SM}}$ denotes the mapping stack. Thus, $\mathfrak{B}(\mathfrak{X})$ is the quotient stack arising from a group action on a stack; see appendix A. The notation $\Pi T\mathfrak{X}$ is motivated by the fact that when \mathfrak{X} is a manifold, the internal hom in question is in fact representable by the parity-reversed tangent bundle. Note that if \mathfrak{X} is an orbifold, that is, is represented by a proper étale Lie groupoid, then so is $\Pi T\mathfrak{X}$.

Theorem 2. *For any differentiable stack \mathfrak{X} , there is a natural bijection*

$$0|1\text{-EFT}(\mathfrak{X}) \cong \Omega_{\text{cl}}^{\text{ev}}(\mathfrak{X})$$

between $0|1\text{-EFTs}$ over \mathfrak{X} and closed differential forms of even parity. If \mathfrak{X} is an orbifold, passing to concordance classes gives an isomorphism with even de Rham cohomology:

$$0|1\text{-EFT}(\mathfrak{X})/\text{concordance} \cong H_{\text{dR}}^{\text{ev}}(\mathfrak{X}).$$

Differential forms and cohomology classes of odd degree are similarly related to field theories twisted by the basic twist \mathcal{T}_1 of [12, definition 6.2].

This is, of course, just a stacky version of theorem 1 in Hohnhold et al. [12], and the proof follows from combining their result with our proposition 8. Since we will consider more general twists in a future paper (or, more specifically, give a classification in terms of flat superconnections, see [15, chapter 4]), we will just sketch the proof when $\mathfrak{X} = X//G$ is a global quotient orbifold here. In that case, $\mathfrak{B}(\mathfrak{X}) \cong \Pi TX // (\text{Isom}(\mathbb{R}^{0|1}) \times G)$ and

$$\begin{aligned} 0|1\text{-EFT}(X//G) &= C^\infty(\Pi TX)^{\text{Isom}(\mathbb{R}^{0|1}) \times G} \\ &= \left(C^\infty(\Pi TX)^{\text{Isom}(\mathbb{R}^{0|1})} \right)^G = (\Omega_{\text{cl}}^{\text{ev}}(X))^G = \Omega_{\text{cl}}^{\text{ev}}(X//G). \end{aligned}$$

2.2. Dimension $1|1$. Denote by $\mathfrak{K}(\mathfrak{X})$ the stack of closed, connected Euclidean $1|1$ -manifolds over \mathfrak{X} : an object (K, ψ) of $\mathfrak{K}(\mathfrak{X})$ over S is given by an S -family K of Euclidean supercircles together with a map $\psi: K \rightarrow \mathfrak{X}$, and a morphism $(K', \psi') \rightarrow (K, \psi)$ over a map $f: S' \rightarrow S$ is given by a fiberwise isometry $F: K' \rightarrow K$ covering f together with a 2-morphism $\psi' \rightarrow \psi \circ F$; compositions are performed in the obvious way.

Any $1|1\text{-EFT}$ over \mathfrak{X} determines a smooth function on $\mathfrak{K}(\mathfrak{X})$, the *partition function* of the theory. This is an immediate consequence of the fact that the empty manifold,

being the monoidal unit in the bordism category, is required to map to the vector space \mathbb{C} .

Finally, let us examine more closely the stack $\mathfrak{K} = \mathfrak{K}(\text{pt})$ of closed connected 1|1-dimensional Euclidean manifolds. Given a parameter supermanifold S and a map $l: S \rightarrow \mathbb{R}_{>0}^{1|1}$, we can form the S -family of Euclidean supercircles of length l , $K_l = (S \times \mathbb{R}^{1|1})/\mathbb{Z}l$, where the generator of the \mathbb{Z} -action is the isometry described, in terms of T -points of $S \times \mathbb{R}^{1|1}$, by $(s, u) \mapsto (s, l(s) \cdot u)$. Moreover, given any map $r: S \rightarrow \mathbb{R}^{1|1}$, the isometry of $S \times \mathbb{R}^{1|1}$ given by $(s, u) \mapsto (s, r(s) \cdot u)$ descends to an isometry $K_{r^{-1}lr} \rightarrow K_l$, and the flip $\text{fl}: \mathbb{R}^{1|1} \rightarrow \mathbb{R}^{1|1}$ (the diffeomorphism negating the odd coordinate) descends to an isometry $K_{\text{fl}(l)} \rightarrow K_l$, since fl is a group automorphism of $\mathbb{R}^{1|1}$.

Note that the right action of $\mathbb{R}^{1|1}$ on itself by conjugation extends to an action of the semidirect product $\mathbb{Z}/2 \ltimes \mathbb{R}^{1|1}$, where $\mathbb{Z}/2$ acts via fl . The above paragraph can be interpreted as the description of a map of stacks $\mathbb{R}_{>0}^{1|1}/(\mathbb{Z}/2 \ltimes \mathbb{R}^{1|1}) \rightarrow \mathfrak{K}$. This map only fails to be an equivalence because the S -point $(0, l)$ of $\mathbb{Z}/2 \ltimes \mathbb{R}^{1|1}$ corresponds to the identity of K_l . See [15, section 3.3.2] for more details.

3. DIMENSIONAL REDUCTION

Since 0|1-dimensional bordism categories have $\{\emptyset\}$ as their stack of objects, it suffices to discuss the functors (1) of internal categories in terms of the corresponding substacks of (fiberwise) closed and connected families of bordisms; slightly abusively, we still call those “bordism stacks”. The two middle bordism stacks, as well as the maps

$$\mathfrak{B}(\Lambda\mathfrak{X}) \leftarrow \mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X}) \rightarrow \mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\Lambda\mathfrak{X}) \rightarrow \mathfrak{K}(\mathfrak{X}).$$

relating them, will be defined in the ensuing subsections. The first map from left to right is full and essentially surjective, and thus induces bijections on sections of sheaves (and exists for an arbitrary stack \mathfrak{Y} in place of $\Lambda\mathfrak{X}$). The corresponding statement for field theories is that we have a bijection between the sets of (twisted) field theories with related twists. The second map is an equivalence, but its inverse does not seem to have a nice geometric description. The third map admits a simple and natural geometric description. We will refer to the two middle stacks as the stacks of \mathbb{T} -equivariant and \mathbb{R}/\mathbb{Z} -equivariant bordisms over $\Lambda\mathfrak{X}$. At the end of this section, we specialize these constructions to the case where $\mathfrak{X} = X//G$ is a global quotient by a finite group, which will hopefully illustrate the ideas.

Following the physical (and, by now, mathematical) jargon, restriction of 1|1-EFTs (or just functions on $\mathfrak{K}(\mathfrak{X})$) to 0|1-EFTs via the above maps of bordism stacks will be referred to as dimensional reduction. Our motivation for doing this is that the stack $\mathfrak{K}(\mathfrak{X})$ of Euclidean supercircles over \mathfrak{X} is “infinite dimensional”, and therefore unwieldy to analysis; dimensional reduction allows us to probe its geometry by means of 0|1-dimensional gadgets over \mathfrak{X} .

3.1. \mathbb{R}/\mathbb{Z} -equivariant bordisms. The inertia $\Lambda\mathfrak{X}$ of a stack \mathfrak{X} in SM is naturally endowed with a pt/\mathbb{Z} -action, and hence an induced \mathbb{R}/\mathbb{Z} -action. We define a stack $\mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\Lambda\mathfrak{X})$ where an object over S is given by the following data:

- (1) a family $\Sigma \rightarrow S$ of connected Euclidean 0|1-manifolds,
- (2) a principal \mathbb{T} -bundle $P \rightarrow \Sigma$ with a fiberwise connection ω whose curvature agrees with the tautological (fiberwise) 2-form $d\zeta$ on Σ (see appendix B), and

- (3) an \mathbb{R}/\mathbb{Z} -equivariant map $\psi: P \rightarrow \Lambda\mathfrak{X}$, with equivariance datum ρ , where \mathbb{R}/\mathbb{Z} acts on P via the usual homomorphism $\mathbb{R}/\mathbb{Z} \rightarrow \mathbb{T}$.

(Recall that \mathbb{R}/\mathbb{Z} -equivariance is not just a condition on ψ , but rather extra data encoded by the 2-morphism ρ , see appendix A). We will usually denote this object (Σ, P, ψ, ρ) or, diagrammatically,

$$\begin{array}{ccc} P & \xrightarrow[\mathbb{R}/\mathbb{Z}]{\psi} & \Lambda\mathfrak{X} \\ \downarrow & & \\ \Sigma. & & \end{array}$$

A morphism $(\Sigma', P', \psi', \rho') \rightarrow (\Sigma, P, \psi, \rho)$ covering a map of supermanifolds $S' \rightarrow S$ is given by

- (1) a fiberwise isometry $F: \Sigma' \rightarrow \Sigma$ covering $S' \rightarrow S$,
- (2) a connection-preserving bundle map $\Phi: P' \rightarrow P$ covering F , and
- (3) an equivariant 2-morphism $\xi: \psi' \rightarrow \psi \circ \Phi$.

Compositions are performed as suggested by the geometry.

Now we discuss the map $\iota: \mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\Lambda\mathfrak{X}) \rightarrow \mathfrak{K}(\mathfrak{X})$. An object (Σ, P, ψ, ρ) over S is mapped to the supercircle over \mathfrak{X} consisting of

- (1) the family of 1|1-dimensional manifolds P endowed with the fiberwise Euclidean structure determined by ω (see theorem 9), and
- (2) the map $P \rightarrow \mathfrak{X}$ obtained by composing ψ with the forgetful map $\Lambda\mathfrak{X} \rightarrow \mathfrak{X}$.

Notice that this construction forgets the \mathbb{T} -action on P as well as the equivariance datum ρ . To define ι at the level of morphisms, recall that a connection-preserving bundle map $P' \rightarrow P$ covering a fiberwise isometry $\Sigma' \rightarrow \Sigma$ is a fiberwise (over S) isometry with respect to the Euclidean structures on P', P .

3.2. \mathbb{T} -equivariant bordisms. For any stack \mathfrak{X} , we define $\mathfrak{B}^{\mathbb{T}}(\mathfrak{X})$ to be the stack whose S -points are given by an S -family of connected Euclidean 0|1-manifolds $\Sigma \rightarrow S$ together with two pieces of data:

- (1) a principal \mathbb{T} -bundle $P \rightarrow \Sigma$ with a fiberwise connection ω whose curvature agrees with the tautological 2-form on Σ ,
- (2) a map $\psi: \Sigma \rightarrow \mathfrak{X}$.

Morphisms between two objects (Σ', P', ψ') and (Σ, P, ψ) over $f: S' \rightarrow S$ consist of a fiberwise isometry $F: \Sigma' \rightarrow \Sigma$ covering f , a connection-preserving bundle map $\Phi: P' \rightarrow P$ covering F , and a 2-morphism $\xi: \psi' \rightarrow \psi \circ F$. Compositions are performed as suggested by the geometry. The data (1) and (2) are completely unrelated in the sense that

$$\mathfrak{B}^{\mathbb{T}}(\mathfrak{X}) \cong \mathfrak{B}^{\mathbb{T}} \times_{\mathfrak{B}} \mathfrak{B}(\mathfrak{X}),$$

and in particular the map $\mathfrak{B}^{\mathbb{T}}(\mathfrak{X}) \rightarrow \mathfrak{B}(\mathfrak{X})$ we are interested in is simply the projection onto the second component. Our interest in $\mathfrak{B}^{\mathbb{T}}(\mathfrak{X})$ is due to the fact that it admits a straightforward quotient stack presentation. Write $\mathbb{T}^{1|1} = \mathbb{R}^{1|1}/\mathbb{Z}$ for the (length 1) super circle group.

Proposition 3. *There is an equivalence of stacks*

$$\Pi T\mathfrak{X} // \text{Isom}(\mathbb{T}^{1|1}) \rightarrow \mathfrak{B}^{\mathbb{T}}(\mathfrak{X}),$$

where the action of $\text{Isom}(\mathbb{T}^{1|1})$ on $\Pi T\mathfrak{X}$ is through the quotient

$$\pi: \text{Isom}(\mathbb{T}^{1|1}) = \mathbb{T}^{1|1} \rtimes \mathbb{Z}/2 \rightarrow \mathbb{R}^{0|1} \rtimes \mathbb{Z}/2 = \text{Isom}(\mathbb{R}^{0|1}).$$

Proof. Given $f: S \rightarrow \Pi T\mathfrak{X}$, we construct an S -point of $\mathfrak{B}^T(\mathfrak{X})$ by letting Σ and P be trivial bundles, the connection form on P be the standard one, namely $\omega = dt - \theta d\theta$, and $\psi: \Sigma = S \times \mathbb{R}^{0|1} \rightarrow \mathfrak{X}$ be the adjoint to f . A morphism $\xi: f \rightarrow f'$ in $(\Pi T\mathfrak{X})_S$ prescribes a morphism in $\mathfrak{B}^T(\mathfrak{X})$ in the obvious way, and this determines a map $F: \Pi T\mathfrak{X} \rightarrow \mathfrak{B}^T(\mathfrak{X})$.

Next, we build a 2-morphism between the two compositions

$$\Pi T\mathfrak{X} \times \text{Isom}(\mathbb{T}^{1|1}) \rightrightarrows \Pi T\mathfrak{X} \rightarrow \mathfrak{B}^T(\mathfrak{X}).$$

Given $f \in \Pi T\mathfrak{X}_S$ as above and $\Phi \in \text{Isom}(\mathbb{T}^{1|1})_S$, we get a diagram

$$\begin{array}{ccccc} S \times \mathbb{T}^{1|1} & \xrightarrow{\Phi} & S \times \mathbb{T}^{1|1} & & \\ \downarrow & & \downarrow & & \\ S \times \mathbb{R}^{0|1} & \xrightarrow{\pi(\Phi)} & S \times \mathbb{R}^{0|1} & \xrightarrow{\psi} & \mathfrak{X} \\ \downarrow & & \downarrow & & \\ S & \xlongequal{\quad} & S & & \end{array}$$

Ignoring the left column we get the data of $F(f)$, and ignoring the right column we get the data of $F(f \cdot \Phi)$; the diagram gives us a morphism between them. Naturality in f and S is clear, and so are the additional compatibility conditions that we need in order to specify an object in the 2-limit of

$$\text{Fun}_{\text{SM}}(\Pi T\mathfrak{X}, \mathfrak{B}^T(\mathfrak{X})) \rightrightarrows \text{Fun}_{\text{SM}}(\Pi T\mathfrak{X} \times \text{Isom}(\mathbb{T}^{1|1}), \mathfrak{B}^T(\mathfrak{X})) \rightrightarrows \dots$$

Using proposition 8, this specifies the map $\Pi T\mathfrak{X} // \text{Isom}(\mathbb{T}^{1|1}) \rightarrow \mathfrak{B}^T(\mathfrak{X})$.

For contractible S , it is clear that any object in $\mathfrak{B}^T(\mathfrak{X})_S$ is equivalent to the one associated to some f as above, so our map of stacks is essentially surjective. Moreover, any morphism in $\mathfrak{B}^T(\mathfrak{X})$ between the images of f, f' is uniquely prescribed by some $\xi: f \rightarrow f'$ and Φ as above, so the map is full and faithful. \square

Under the identification of the proposition, the map $\mathfrak{B}^T(\mathfrak{X}) \rightarrow \mathfrak{B}(\mathfrak{X})$ becomes the natural map

$$\Pi T\mathfrak{X} // (\mathbb{T}^{1|1} \rtimes \mathbb{Z}/2) \rightarrow \Pi T\mathfrak{X} // (\mathbb{R}^{0|1} \rtimes \mathbb{Z}/2)$$

induced by the quotient $\pi: \mathbb{T}^{1|1} \rtimes \mathbb{Z}/2 \rightarrow \mathbb{R}^{0|1} \rtimes \mathbb{Z}/2$. Since the action on $\Pi T\mathfrak{X}$ factors through π , it follows that this map induces an isomorphism on functions, and more generally on global sections of any sheaf on SM.

3.3. The map $\mathfrak{B}^T(\Lambda\mathfrak{X}) \rightarrow \mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\Lambda\mathfrak{X})$. We denote by α the canonical automorphism of the identity of $\Lambda\mathfrak{X}$. It suffices to describe the restriction of the desired map $\mathfrak{B}^T(\Lambda\mathfrak{X}) \rightarrow \mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\Lambda\mathfrak{X})$ to the full prestack of objects where all bundles involved are trivial. To $(\Sigma, P, \psi) \in \mathfrak{B}^T(\Lambda\mathfrak{X})_S$ with

$$\Sigma = S \times \mathbb{R}^{0|1}, \quad P = S \times \mathbb{T}^{1|1}, \quad \psi: \Sigma \rightarrow \Lambda\mathfrak{X},$$

and the standard Euclidean structure and connection form, we want to assign an object $(\Sigma, P, \psi_1: P \rightarrow \Lambda\mathfrak{X}, \rho) \in \mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\mathfrak{X})_S$. Consider the covering $U = S \times \mathbb{R}^{1|1} \rightarrow P$. Our goal is to descend $\tilde{\psi}: U \rightarrow \Lambda\mathfrak{X}$, the pullback of ψ via $U \rightarrow \Sigma$, to a map $\psi_1: P \rightarrow \Lambda\mathfrak{X}$.

$$\begin{array}{ccccc} & \tilde{\psi} & & & \\ & \curvearrowright & & & \\ U & \longrightarrow & P & \longrightarrow & \Sigma \xrightarrow{\psi} \Lambda\mathfrak{X} \\ & & \psi_1 & \dashrightarrow & \uparrow \\ & & & & \end{array}$$

In order to do that, we need to provide certain isomorphisms over double overlaps and then check a coherence condition on triple overlaps. Denote by $\text{pr}_1, \text{pr}_2: U \times_P U \rightrightarrows U$ the projections. Then we are looking for a 2-morphism $\tilde{\alpha}: \tilde{\psi} \circ \text{pr}_1 \rightarrow \tilde{\psi} \circ \text{pr}_2$ (whose domain and codomain happen to be the same map, henceforth denoted $\psi \circ \text{pr}$). Note that $U \times_P U$ breaks up as a disjoint union indexed by \mathbb{Z} , where the n th component comprises pairs of the form $(x, n \cdot x)$. On that component, we set $\tilde{\alpha}$ to be the horizontal composition (whiskering)

$$\tilde{\alpha} = \alpha^n \circ (\psi \circ \text{pr}).$$

Regarding the coherence condition, we need to check that

$$(2) \quad \text{pr}_{13}^* \tilde{\alpha} = \text{pr}_{23}^* \tilde{\alpha} \circ \text{pr}_{12}^* \tilde{\alpha},$$

where pr_{ij} denotes the projection $U \times_P U \times_P U \rightarrow U \times_P U$ forgetting the third index. The threefold fiber product breaks up as a disjoint union indexed by $\mathbb{Z} \times \mathbb{Z}$, where the component (n, m) and its image through the pr_{ij} are as follows.

$$\begin{array}{ccccc} & (x, n \cdot x, (n+m) \cdot x) & & & \\ & \swarrow \text{pr}_{12} & \downarrow \text{pr}_{13} & \searrow \text{pr}_{23} & \\ (x, n \cdot x) & & (x, (n+m) \cdot x) & & (n \cdot x, (n+m) \cdot x) \end{array}$$

Therefore, on that component,

$$\begin{aligned} \text{pr}_{23}^* \tilde{\alpha} &= \alpha^m \circ (\psi \circ \text{pr}), & \text{pr}_{12}^* \tilde{\alpha} &= \alpha^n \circ (\psi \circ \text{pr}), \\ \text{pr}_{13}^* \tilde{\alpha} &= \alpha^{n+m} \circ (\psi \circ \text{pr}), \end{aligned}$$

and their vertical compositions are as required by (2). We thus obtain the desired $\psi_!: P \rightarrow \Lambda \mathfrak{X}$.

Next, we need to provide the \mathbb{R}/\mathbb{Z} -equivariance datum ρ for $\psi_!$. To analyze the putative square

$$(3) \quad \begin{array}{ccc} P \times \mathbb{R}/\mathbb{Z} & \xrightarrow{\mu} & P \\ \psi_! \times \text{id} \downarrow & \nearrow \rho & \downarrow \psi_! \\ \Lambda \mathfrak{X} \times \mathbb{R}/\mathbb{Z} & \longrightarrow & \Lambda \mathfrak{X} \end{array}$$

we notice that, after a suitable base change, any S -point of $P \times \mathbb{R}/\mathbb{Z}$ can be pulled back from the atlas $i_0: P \times \mathbb{R} \rightarrow P \times \mathbb{R}/\mathbb{Z}$, or, for that matter, from any of the atlases $i_n: (p, t) \mapsto i_0(p, t+n)$, where $n \in \mathbb{Z}$; moreover, any morphism of S -points can be pulled back from $m: i_n \rightarrow i_{n+m}$. Thus, we can extract all information encoded by ρ by evaluating the above diagram on each i_n and m . The top-right composition factors through $P \times \mathbb{T}$, so every i_n maps to the same $\mu^* \psi_! \in \Lambda \mathfrak{X}_{P \times \mathbb{R}}$, and m maps to the identity. The left-bottom composition factors through $\Lambda \mathfrak{X} \times \text{pt}/\mathbb{Z}$, so, for any n , i_n maps to $\text{pr}_1^* \psi_! \in \Lambda \mathfrak{X}_{P \times \mathbb{R}}$, and $m: i_n \rightarrow i_{n+m}$ maps to $\text{pr}_1^* \alpha^m: \text{pr}_1^* \psi_! \rightarrow \text{pr}_1^* \psi_!$. For each i_n , the fibered natural transformation ρ should give a morphism $\rho(i_n): \text{pr}_1^* \psi_! \rightarrow \mu^* \psi_!$ fitting in the diagram below.

$$\begin{array}{ccc} \text{pr}_1^* \psi_! & \xrightarrow{\rho(i_n)} & \mu^* \psi_! \\ \text{pr}_1^* \alpha^m \downarrow & & \parallel \\ \text{pr}_1^* \psi_! & \xrightarrow{\rho(i_{n+m})} & \mu^* \psi_! \end{array}$$

This means ρ is completely specified by $\rho(i_0)$, and naturality imposes no further restrictions on the latter. To provide $\rho(i_0)$, it suffices to give a morphism $\mathrm{pr}_1^* \tilde{\psi} \rightarrow \mu^* \tilde{\psi}$, where the latter is the composition

$$U \times \mathbb{R} \rightarrow P \times \mathbb{R} \xrightarrow{\mu} P \rightarrow \Sigma \xrightarrow{\psi} \Lambda \mathfrak{X},$$

satisfying appropriate coherence conditions on $U \times_P U \times \mathbb{R}$. Since $\mu^* \tilde{\psi} = \mathrm{pr}_1^* \tilde{\psi}$, we can take that to be the identity. One can check that ρ satisfies the coherence conditions required of the equivariance datum.

The effect of $\mathfrak{B}^{\mathbb{T}}(\Lambda \mathfrak{X}) \rightarrow \mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\Lambda \mathfrak{X})$ on morphisms is also given by descent. Given a morphism in $\mathfrak{B}^{\mathbb{T}}(\Lambda \mathfrak{X})$

$$\begin{array}{ccc} P & \xrightarrow{\Phi} & P' \\ \downarrow & \nearrow F & \downarrow \psi \\ \Sigma & \xrightarrow{\quad} & \Sigma' \end{array} \quad \begin{array}{c} \nearrow \xi \\ \searrow \psi' \end{array} \quad \Lambda \mathfrak{X}$$

where $\Sigma' = S' \times \mathbb{R}^{0|1}$, $P' = S' \times \mathbb{T}^{1|1}$ are also trivial families, consider the fiberwise universal cover $U' = S \times \mathbb{R}^{1|1} \rightarrow P'$ and choose a lift $\tilde{\Phi}: U \rightarrow U'$. We can then lift ψ , ψ' and ξ by composing respectively whiskering with $U \rightarrow \Sigma$ or $U' \rightarrow \Sigma'$

$$\begin{array}{ccc} U & \xrightarrow{\tilde{\Phi}} & U' \\ \searrow \tilde{\psi} & \xrightarrow{\tilde{\xi}} & \nearrow \tilde{\psi}' \\ & \Lambda \mathfrak{X} & \end{array}$$

and descend $\tilde{\xi}$ to a morphism $\xi_! : \psi_! \rightarrow \Phi^* \psi'_!$. To justify that, we need to show that on the n th component of $U \times_P U$ the diagram

$$\begin{array}{ccc} \mathrm{pr}_1^* \tilde{\psi} & \xrightarrow{\mathrm{pr}_1^* \tilde{\xi}} & \mathrm{pr}_1^* (\tilde{\psi}' \circ \tilde{\Phi}) \\ \alpha^n \downarrow & & \downarrow \alpha^n \\ \mathrm{pr}_2^* \tilde{\psi} & \xrightarrow{\mathrm{pr}_2^* \tilde{\xi}} & \mathrm{pr}_2^* (\tilde{\psi}' \circ \tilde{\Phi}) \end{array}$$

commutes. (To be precise, α^n above stands, respectively, for $\alpha^n \circ (\psi \circ \mathrm{pr})$, the gluing isomorphism used to build $\psi_!$, and its counterpart for $\Phi^* \psi'_!$.) This follows immediately from the compatibility condition between ξ and α , namely $\xi \circ \alpha_\psi = \alpha_{\Phi^* \psi'} \circ \xi$. The morphism $\xi_!$ thus obtained is independent of the choice of lift $\tilde{\Phi}$, since it only depends on the composition $\tilde{\psi}' \circ \tilde{\Phi}$. We omit the verification that $\xi_!$ is compatible with the equivariance data.

Finally, we assign to the morphism in $\mathfrak{B}^{\mathbb{T}}(\Lambda \mathfrak{X})$ prescribed by the data (F, Φ, ξ) the morphism in $\mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\Lambda \mathfrak{X})$ prescribed by the data $(F, \Phi, \xi_!)$. That this assignment respects compositions follows from uniqueness for descent of morphisms.

Theorem 4. *The fibered functor $\mathfrak{B}^{\mathbb{T}}(\Lambda \mathfrak{X}) \rightarrow \mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\Lambda \mathfrak{X})$ is an equivalence.*

Proof. At the morphism level, the effect of the functor in question was described in two steps: $\xi \mapsto \tilde{\xi} \mapsto \xi_!$. This is a one-to-one procedure because the first step is injective (since $U \rightarrow \Sigma$ has local sections) and the second step (descent) is in fact bijective. Thus, it remains to show that the fibered functor $\mathfrak{B}^{\mathbb{T}}(\Lambda \mathfrak{X}) \rightarrow \mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\Lambda \mathfrak{X})$ is full and essentially surjective. In order to do that, we will build a prestack $\mathfrak{B}^{\mathrm{triv}}$

and a factorization

$$(4) \quad \begin{array}{ccc} & \mathfrak{B}^{\text{triv}} & \\ v \swarrow & & \searrow u \\ \mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X}) & \longrightarrow & \mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\Lambda\mathfrak{X}). \end{array}$$

where u is full and essentially surjective on the groupoid of S -point for any contractible S .

The prestack $\mathfrak{B}^{\text{triv}}$ is defined as follows:

- (1) an object consists of an object $(\Sigma, P, \psi, \rho) \in \mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\Lambda\mathfrak{X})$ together with a section $s: \Sigma \rightarrow P$, and
- (2) a morphism $(\Sigma', P', \psi', \rho', s') \rightarrow (\Sigma, P, \psi, \rho, s)$ is a pair consisting of a morphism (F, Φ, ξ) of the underlying objects in $\mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\mathfrak{X})$ together with a map $r: \Sigma' \rightarrow \mathbb{R}$ relating s and s' in the sense that $\Phi \circ s' = (s \circ F)e^{2\pi i r}$.

With a little poetic license, a morphism can be depicted as follows (the square containing r would literally make sense, as a 2-commutative diagram, if we replaced P with $P//\mathbb{R}$).

$$(5) \quad \begin{array}{ccccc} & & \psi' & & \\ & & \searrow & & \\ P' & \xrightarrow{\Phi} & P & \xrightarrow{\psi} & \Lambda\mathfrak{X} \\ \uparrow s' & \searrow r & \uparrow s & & \\ \Sigma' & \xrightarrow{F} & \Sigma & & \end{array}$$

We define $u: \mathfrak{B}^{\text{triv}} \rightarrow \mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\Lambda\mathfrak{X})$ to be the forgetful functor, which simply discards s and r , so it is clearly full and essentially surjective over contractible S as claimed.

Next, we construct $v: \mathfrak{B}^{\text{triv}} \rightarrow \mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X})$. To an object $(\Sigma, P, \psi, \rho, s) \in \mathfrak{B}^{\text{triv}}$, we assign the object $(\Sigma, P, s^*\psi)$ in $\mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X})$. Now fix a morphism as in (5). To define its image in $\mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X})$, the only new data we need to provide is a morphism $(s')^*\psi' \rightarrow (s \circ F)^*\psi$, which we take to be the following composition:

$$(s')^*\psi' \xrightarrow{(s')^*\xi} (s')^*\Phi^*\psi \cong (\Phi \circ s')^*\psi = ((s \circ F)e^{2\pi i r})^*\psi \xrightarrow{\rho_{s \circ F, r}^{-1}} (s \circ F)^*\psi.$$

We omit the verification of functoriality.

To finish the proof, we just need to show that (4) commutes (up to 2-isomorphism). It suffices to look at $(\Sigma, P, \psi, \rho, s) \in \mathfrak{B}^{\text{triv}}$ where P and Σ are trivial families, and pick s to be the unit section; our goal is to produce an isomorphism between $(s^*\psi)_!$ and ψ , natural in the input data $(\Sigma, P, \psi, \rho, s)$ and compatible with the respective equivariance data. From the discussion leading to the construction of the ρ in (3), we see that the data of the present (arbitrarily given) ρ is essentially an isomorphism $\rho_0: \text{pr}_1^*\psi \rightarrow \mu^*\psi$ in $\Lambda\mathfrak{X}_{P \times \mathbb{R}}$. Now, let $\pi^*\psi$ be the pullback through $\pi: U \rightarrow P$ and recall that $\widetilde{s^*\psi}$ is the U -point of $\Lambda\mathfrak{X}$ used to put together $(s^*\psi)_!$. Note that each half of the diagram

$$\begin{array}{ccccc} & & \widetilde{s^*\psi} & & \\ & \searrow & & \searrow & \\ U = \Sigma \times \mathbb{R} & \xrightarrow{s \times \text{id}} & P \times \mathbb{R} & \xrightarrow{\text{pr}_1} & P \xrightarrow{\psi} \Lambda\mathfrak{X} \\ & \searrow & \mu & \searrow & \\ & & \pi^*\psi & & \end{array}$$

commutes, so ρ_0 gives a morphism $\widetilde{s^*\psi} \rightarrow \pi^*\psi$ and, by descent, a morphism $(s^*\psi)_! \rightarrow \psi$. We omit the naturality and compatibility checks. \square

3.4. Global quotients. Let us illustrate the above constructions when $\mathfrak{X} = X//G$ is the quotient orbifold associated to the action of a finite group G on a manifold X .

We start noticing that a quotient stack presentation for $\Lambda(X//G)$ can be given as follows. Consider the product $X \times G$ with diagonal G -action, where G acts on itself by conjugation. There is an invariant submanifold

$$\hat{X} = \{(x, g) \in X \times G \mid x \in X^g\},$$

and an object over S in the quotient stack $\hat{X}//G$ consists of a pair $(Q, (f, A))$, where $Q \rightarrow S$ is a principal G -bundle and $(f, A): Q \rightarrow \hat{X} \subset X \times G$ is a G -equivariant smooth map. Denote by $\alpha: Q \rightarrow Q$ the bundle automorphism determined by A ; on T -points, it is given by

$$\alpha(q) = qA(q), \quad q \in Q_T.$$

Notice that this automorphism preserves f , and therefore (Q, f, α) determines an S -point of $\Lambda(X//G)$. Conversely, given an S -point (Q, f, α) of $\Lambda(X//G)$, we can specify a G -equivariant map $A: Q \rightarrow G$ by requiring that the above equation holds, and compatibility between f and α implies that the resulting map $(f, A): Q \rightarrow X \times G$ factors through \hat{X} , thus determining an object of $\hat{X}//G$ over S .

The translation back and forth between A and α provides a pt/\mathbb{Z} -equivariant equivalence between $\Lambda(X//G)$ and $\hat{X}//G$, compatible with the maps $\Lambda(X//G) \rightarrow X//G$ forgetting the prescribed automorphism and $\hat{X}//G \rightarrow X//G$ induced the projection $\text{pr}_1: X \times G \rightarrow X$. We will shift freely between these two formulations.

The geometric content of an S -family in $\mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\hat{X}//G)$ is the following:

- (1) a family $\Sigma \rightarrow S$ of connected Euclidean 0|1-manifolds,
- (2) a principal \mathbb{T} -bundle $P \rightarrow \Sigma$ with a fiberwise connection ω whose curvature agrees with the tautological 2-form on Σ ,
- (3) a principal G -bundle $Q \rightarrow P$,
- (4) a G -equivariant map $(f, A): Q \rightarrow \hat{X} \subset X \times G$; or, equivalently, a bundle automorphism $\alpha: Q \rightarrow Q$ and a G -equivariant map $f: Q \rightarrow X$ such that $f \circ \alpha = f$, and, finally,
- (5) a collection of natural isomorphisms of G -torsors

$$\rho_{p,t}: Q_p \rightarrow Q_{pe^{2\pi i t}}$$

for each pair of T -points $q: T \rightarrow P$, $t: T \rightarrow \mathbb{R}$, intertwining the maps

$$f_p: Q_p \rightarrow X, \quad f_{pe^{2\pi i t}}: Q_{pe^{2\pi i t}} \rightarrow X$$

and subject to the condition that for any $n: T \rightarrow \mathbb{Z}$ the diagram

$$(6) \quad \begin{array}{ccc} Q_p & \xrightarrow{\rho_{p,t}} & Q_{pe^{2\pi i t}} \\ \alpha_p^n \downarrow & & \parallel \\ Q_p & \xrightarrow{\rho_{p,t+n}} & Q_{pe^{2\pi i(t+n)}} \end{array}$$

commutes.

The last condition means that α agrees with the holonomy of Q around the fibers of P . A morphism in $\mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\hat{X}//G)$ is given by a fiberwise isometry $F: \Sigma' \rightarrow \Sigma$, a connection-preserving bundle map $\Phi: P' \rightarrow P$ covering F , and a bundle map

$Q' \rightarrow Q$ covering Φ which is required to be compatible in the obvious way with the data in (4) and (5) above.

The geometric content of an S -family in $\mathfrak{B}^{\mathbb{T}}(\hat{X} // G)$ is the following:

- (1) a family $\Sigma \rightarrow S$ of connected Euclidean 0|1-manifolds,
- (2) a principal \mathbb{T} -bundle $P \rightarrow \Sigma$ with a connection ω whose curvature agrees with the tautological 2-form on Σ ,
- (3) a principal G -bundle $Q \rightarrow \Sigma$, and
- (4) a G -equivariant map $f: Q \rightarrow \hat{X}$.

A morphism $(\Sigma', P', Q', f') \rightarrow (\Sigma, P, Q, f)$ consists of a fiberwise isometry $F: \Sigma' \rightarrow \Sigma$, a connection-preserving bundle map $\Phi: P' \rightarrow P$ covering F , and a bundle map $Q' \rightarrow Q$ covering F and intertwining the maps $f: Q \rightarrow \hat{X}$ and $f': Q' \rightarrow \hat{X}$. From proposition 3, it follows that $\mathfrak{B}^{\mathbb{T}}(\hat{X} // G)$ admits the presentation

$$(\Pi T(\hat{X} // G)) // \text{Isom}(\mathbb{T}^{1|1}) \cong \Pi T \hat{X} // (\text{Isom}(\mathbb{T}^{1|1}) \times G).$$

Finally, let us describe the map relating \mathbb{T} -equivariant and $\mathbb{R} // \mathbb{Z}$ -equivariant bordisms in this special situation. Fix $(\Sigma, P, Q, f) \in \mathfrak{B}^{\mathbb{T}}(\hat{X} // G)_S$ and let $(\Sigma, P, Q!, f!, \rho) \in \mathfrak{B}^{\mathbb{R} // \mathbb{Z}}(\hat{X} // G)_S$ be its image. Locally in S , f determines a conjugacy class of G and $Q! \rightarrow P$ is the G -bundle with that holonomy around the fibers of $P \rightarrow \Sigma$. More specifically, let us assume P and Q are trivial; if S is connected, then f determines an element $g \in G$, namely the one corresponding to the connected component of $\hat{X} = \coprod_{g \in G} X^g$ in which $f|_{\Sigma \times \{e\}}$ takes values. Then $Q! \rightarrow P$ is the G -bundle built as a quotient

$$Q! = (\Sigma \times \mathbb{R} \times G) / \mathbb{Z} \rightarrow P = \Sigma \times \mathbb{T},$$

where the \mathbb{Z} -action is generated by the diffeomorphism prescribed, on T -points, by $(s, t, h) \mapsto (s, t + 1, gh)$. The map $f!: Q! \rightarrow \hat{X}$ is induced by the \mathbb{Z} -invariant map $(s, t, h) \mapsto f(s, e) \cdot h$. The automorphism of $Q!$ determined by the G -component of $f!$ can be expressed as $(s, t, h) \mapsto (s, t, gh)$.

3.5. More general kinds of equivariant bordisms. Using the $B\mathbb{T}$ -action on \mathfrak{K} (i.e., the natural \mathbb{T} -action attached to every family of Euclidean supercircles), we could also associate, to any stack \mathfrak{Y} with a chosen automorphism, bordism stacks $\mathfrak{K}^{\mathbb{T}}(\mathfrak{Y})$ and $\mathfrak{K}^{\mathbb{R} // \mathbb{Z}}(\mathfrak{Y})$ by imposing, on each map $\mathfrak{K} \ni K \rightarrow \mathfrak{Y}$, the equivariance condition indicated by the superscript; there are also variants based on \mathfrak{K}_{ev} , the stack of supercircles of purely even length. In each case, we would still have comparison maps as above. Since our goal here is to probe $\mathfrak{K}(\mathfrak{X})$ with the simplest possible objects, we only care about the substacks of supercircles of length 1, that is,

$$\mathfrak{K}_1^{\mathbb{T}}(\Lambda \mathfrak{X}) \cong \mathfrak{B}^{\mathbb{T}}(\Lambda \mathfrak{X}), \quad \mathfrak{K}_1^{\mathbb{R} // \mathbb{Z}}(\Lambda \mathfrak{X}) \cong \mathfrak{B}^{\mathbb{R} // \mathbb{Z}}(\Lambda \mathfrak{X}).$$

Berwick-Evans calls $\mathfrak{K}_{\text{ev}}^{\mathbb{T}}(\mathfrak{X})$ the stack of classical vacua over \mathfrak{X} in [4]. One of his key ideas can be interpreted as follows: super loops of purely even length have the special property of admitting a map $\mathfrak{K}_{\text{ev}} \rightarrow \mathfrak{B}$, and there is an identification

$$\mathfrak{K}_{\text{ev}}^{\mathbb{T}}(\mathfrak{X}) \cong \mathfrak{K}_{\text{ev}} \times_{\mathfrak{B}} \mathfrak{B}(\mathfrak{X}).$$

4. THE CHERN CHARACTER FOR GLOBAL QUOTIENTS

In this section, we show how to recover, in terms of dimensional reduction of field theories, the delocalized Chern character of Baum and Connes [1] (and, before them, Słomińska [14]), concerning the case of a finite group G acting on a manifold X .

We start by briefly recalling the classical construction of ch_G in section 4.1. On field theory side, we can associate to each vector bundle with connection V on an orbifold \mathfrak{X} a field theory $E_V \in 1|1\text{-EFT}(\mathfrak{X})$. To make the discussion shorter, we will only describe, in section 4.2, the partition function of this theory, denoted $Z_V \in C^\infty(\mathfrak{K}(\mathfrak{X}))$. Finally, in section 4.3 we prove theorem 1.

4.1. The Baum–Connes Chern character. As before, we write $\hat{X} = \{(x, g) \in X \times G \mid xg = x\} = \coprod_{g \in G} X^g$. The equivariant Chern character is a ring homomorphism

$$(7) \quad \text{ch}_G: K_G^i(X) \rightarrow H_G^i(\hat{X}; \mathbb{C}) \cong \left[\bigoplus_{g \in G} H^i(X^g; \mathbb{C}) \right]^G.$$

Here, $i \in \mathbb{Z}/2$ and ordinary cohomology is $\mathbb{Z}/2$ -graded. We recall that the equivariant ordinary cohomology of \hat{X} with coefficients in \mathbb{C} can be identified with the invariants in its nonequivariant cohomology; this can be deduced from the Serre spectral sequence for the fibration $EG \times_G X \rightarrow BG$ using the fact that the integral reduced cohomology of a finite group is torsion.

For each $g \in G$, we define the homomorphism $\text{ch}_g: K_G^i(X) \rightarrow H^i(X^g; \mathbb{C})$ as the composition

$$K_G^i(X) \rightarrow K_{\langle g \rangle}^i(X^g) \cong K^i(X^g) \otimes R(\langle g \rangle) \xrightarrow{\text{ch} \otimes \text{tr}_g} H^i(X^g; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

(The middle isomorphism is due to the fact that the action of the cyclic group $\langle g \rangle$ generated by g on X^g is trivial; ch denotes the usual, nonequivariant Chern character, and tr_g assigns to any representation of $\langle g \rangle$ the trace of the operator g .) Finally, we let $\text{ch}_G: K_G^i(X) \rightarrow H_G^i(\hat{X}; \mathbb{C})$ be the direct sum of all ch_g via the identification (7).

Concretely, the effect of ch_g on the K -theory class represented by a G -equivariant vector bundle $V \rightarrow X$ is the following. For each $x \in X^g$, the fiber V_x is a representation of the cyclic group generated by g . Let $\lambda_1, \dots, \lambda_r$ be distinct eigenvalues, and V_x^1, \dots, V_x^r the corresponding eigenspaces. Each λ_i is a $|g|$ -root of unity, so $V|_{X^g}$ can be written as direct sum of vector bundles

$$V|_{X^g} = V^1 \oplus \dots \oplus V^r.$$

Then

$$\text{ch}_g(V) = \sum \lambda_i \text{ch}(V^i) \in H^{\text{ev}}(X^g; \mathbb{C})$$

and

$$\text{ch}_G(V) = \bigoplus_{g \in G} \text{ch}_g(V) \in \left[\bigoplus_{g \in G} H^{\text{ev}}(X^g; \mathbb{C}) \right]^G.$$

This is the correct equivariant extension of the Chern character in the sense that, for compact X , it induces an isomorphism after tensoring with \mathbb{C} . Note that, in the light of the Atiyah–Segal completion theorem, the so-called delocalized cohomology ring $H_G^*(\hat{X}; \mathbb{C})$ is a stronger invariant than ordinary equivariant cohomology of X .

4.2. Parallel transport and field theories. Let \mathfrak{X} be a differentiable stack and $V: \mathfrak{X} \rightarrow \text{Vect}^\nabla$ a vector bundle with connection. Then we can construct a field theory $E_V \in 1|1\text{-EFT}(\mathfrak{X})$ using parallel transport along superpaths in \mathfrak{X} . Roughly speaking, this EFT assigns to a superpoint $x: \text{spt} \rightarrow \mathfrak{X}$ the fiber V_x , and to a superinterval $c: I_{t,\theta} \rightarrow X$ the parallel transport map $\text{SP}(c): V_{c(0,0)} \rightarrow V_{c(t,\theta)}$ constructed by Dumitrescu [8]. It is then part of the conjecture of Stolz and Teichner on the relation between 1|1-EFTs and K -theory that, for reasonable \mathfrak{X} , the field theory

above corresponds to the K -theory class represented by V . We note, however, that there are some subtleties that need to be addressed in the above construction. For instance, objects in $1|1\text{-EBord}(\mathfrak{X})$ are not merely $(S\text{-families of})$ superpoints, but rather germs of collars $S \times \text{spt} \subset S \times \mathbb{R}^{1|1}$; in order to make the sheaf of vector spaces on S associated to such germ independent of the chosen representative, we need to consider flat sections on a neighborhood of the origin in $\mathbb{R}^{1|1}$, and then prove that all linear maps arising from bordisms in fact act on the space of germs of flat sections.

In any case, we are presently only interested in the partition function of the above field theory, i.e., the function $Z_V \in C^\infty(\mathfrak{K}(\mathfrak{X}))$ obtained by restricting E_V to the Euclidean super loop stack $\mathfrak{K}(\mathfrak{X})$. (Note that the reduced field theory $\text{red}(E_V)$ relevant for theorem 1 only depends on Z_V .) The partition function admits a straightforward description independently of the details of the construction of the full EFT: to each S -point (K, ψ) of $\mathfrak{K}(\mathfrak{X})$, we associate the supertrace of the holonomy along K . More specifically, identifying $K \cong (S \times \mathbb{R}^{1|1})/\mathbb{Z}l$ for some “length function” $l: S \rightarrow \mathbb{R}_{>0}^{1|1}$ (locally in S , at least), we consider the parallel transport of $\psi^*V \rightarrow K$ along the superpath $S \times \mathbb{R}^{1|1} \rightarrow K$ with endpoints

$$i_0, i_l: S \rightarrow S \times \mathbb{R}^{1|1}, \quad s \mapsto (s, 0) \text{ respectively } (s, l(s))$$

and then take the fiberwise supertrace of that operator, yielding a smooth function on S .

Proposition 5. *For supercircles of the form $K_l = (S \times \mathbb{R}^{1|1})/\mathbb{Z}l$, the above quantity is preserved by isometries, and therefore defines a function $Z_V \in C^\infty(\mathfrak{K}(\mathfrak{X}))$.*

Proof. Super parallel transport is invariant under fiberwise isometries of $\mathbb{R}^{1|1}$ and gluing of superintervals [8, theorem 3.5], and we saw in section 2.2 that, locally in the parameter space, any isometry $F: K_{l'} \rightarrow K_l$ lifts to a fiberwise isometry \tilde{F} of $\mathbb{R}^{1|1}$. Thus, assuming for simplicity that F covers the identity $S \rightarrow S$ and that $0 \leq \tilde{F}(0) < l \leq \tilde{F}(l)$, we have

$$\begin{aligned} Z_V(K_l, \psi) &= \text{str}(\text{SP}(\tilde{F}(0), l) \circ \text{SP}(0, \tilde{F}(0))) \\ &= \text{str}(\text{SP}(l, \tilde{F}(l')) \circ \text{SP}(\tilde{F}(0), l)) \\ &= Z_V(K_{l'}, \psi \circ F). \end{aligned}$$

Here, $\text{SP}(a_0, a_1)$ denotes parallel transport in ψ^*V along the canonical superpath $S \times \mathbb{R}^{1|1} \rightarrow K_l$ between the endpoints $a_0 \leq a_1: S \rightarrow \mathbb{R}^{1|1}$, and we used the fact that $\text{SP}(0, \tilde{F}(0)) = \text{SP}(l, \tilde{F}(l'))$ under the natural identification between the vector bundles in question. \square

4.3. Proof of theorem 1. As before, fix $V: X//G \rightarrow \text{Vect}^\nabla$. This map classifies a G -equivariant vector bundle over X , which we still call V , with a G -invariant connection ∇ . To get started, we need to describe the pullback of V to a supercircle over $X//G$.

Proposition 6. *Fix a supercircle $\psi: K \rightarrow X//G$ and denote by $\pi: Q \rightarrow K$ and $f: Q \rightarrow X$ the principal G -bundle and G -equivariant map classified by ψ . Then*

there is a natural connection-preserving isomorphism of vector bundles

$$\begin{array}{ccc} (f^*V)/G & \longrightarrow & \psi^*V \\ \downarrow & & \downarrow \\ Q/G & \xlongequal{\quad} & K. \end{array}$$

Proof. Consider the diagram

$$\begin{array}{ccccc} Q \times_K Q & \rightrightarrows & Q & \xrightarrow{f} & X \\ \downarrow \pi & & \downarrow \pi & & \downarrow x \\ K & \xrightarrow{\psi} & X//G & \xrightarrow{V} & \text{Vect}^\nabla. \end{array}$$

Here, $x: X \rightarrow X//G$ is the standard atlas and hence $V \circ x$ classifies the vector bundle with connection $V \rightarrow X$. Notice that the square 2-commutes. In fact, the top-right composition $Q \rightarrow X//G$ classifies the trivial G -bundle $Q \times G \rightarrow Q$, while the left-bottom composition classifies the G -bundle $\pi^*Q \rightarrow Q$ (together with the corresponding equivariant maps into X induced by f), and these two Q -points of $X//G$ are isomorphic.

Now, the composition $V \circ x \circ f$ classifies the vector bundle $f^*V \rightarrow Q$, and the G -equivariance information provides descent data for the covering $Q \times_K Q \cong Q \times G \rightrightarrows Q \rightarrow K$. The descended vector bundle with connection can be described explicitly as $(f^*V)/G \rightarrow K$. Thus, 2-commutativity of the square above and the uniqueness property of descent provide a canonical isomorphism $\psi^*V \cong (f^*V)/G$. \square

To identify the image of $Z_V \in C^\infty(\mathfrak{R}(\hat{X}))$ in $C^\infty(\mathfrak{B}(\hat{X}//G))$ via dimensional reduction with a (G -invariant, even, closed) differential form on \hat{X} , following the identifications from section 3.2, we need to consider the versal $\Pi T\hat{X}$ -family in $\mathfrak{B}(\hat{X})$

$$(8) \quad \begin{array}{ccc} \Pi T\hat{X} \times \mathbb{R}^{0|1} & \xrightarrow{\text{ev}} & \hat{X} \\ \downarrow & & \\ \Pi T\hat{X} & & \end{array}$$

and calculate the corresponding smooth function on the parameter manifold $\Pi T\hat{X}$. The counterpart of the above family in $\mathfrak{B}^\mathbb{T}(\hat{X}//G)$ is obtained by adding the trivial principal \mathbb{T} -bundle with standard connection over $\Pi T\hat{X} \times \mathbb{R}^{0|1}$ and replacing the map into \hat{X} with its postcomposition with the atlas $\hat{x}: \hat{X} \rightarrow \hat{X}//G$. In turn, the image of that gadget in $\mathfrak{B}^{\mathbb{R}/\mathbb{Z}}(\hat{X}//G)$, once restricted to $\Pi TX^g \subset \Pi T\hat{X}$, comprises the following data:

- (1) the family of Euclidean $0|1$ -manifolds $\Sigma = \Pi TX^g \times \mathbb{R}^{0|1} \rightarrow \Pi TX^g$,
- (2) the trivial \mathbb{T} -bundle $P = \Pi TX^g \times \mathbb{R}^{0|1} \times \mathbb{T} \rightarrow \Sigma$, with the standard connection form $\omega = dt - \theta d\theta$,
- (3) the principal G -bundle $Q = (\Pi TX^g \times \mathbb{R}^{1|1} \times G)/\mathbb{Z} \rightarrow P$, where the \mathbb{Z} -action is generated by the map described on S -points by

$$(x, t, h) \in (\Pi TX^g \times \mathbb{R}^{1|1} \times G)_S \mapsto (x, 1 \cdot t, gh),$$

- (4) the map $f: Q \rightarrow \hat{X} \subset X \times G$ given by $(x, t, h) \mapsto (\text{ev}(t_1, x) \cdot h, h^{-1}gh)$, which is well defined since $\text{ev}(t_1, x)$ lies in X^g .

Finally, by proposition 6, the image of the above object in $\mathfrak{K}(X//G)$ can be identified with the ΠTX^g -family of length 1 supercicles $K = \Pi TX^g \times \mathbb{T}^{1|1}$ together with the vector bundle with connection $W = (f^*V)/G \rightarrow K$.

Our task now is to compute the supertrace of the holonomy around K . More precisely, consider the standard superpath $c: \Pi TX^g \times \mathbb{R}^{1|1} \rightarrow \Pi TX^g \times \mathbb{T}^{1|1}$ with endpoints $i_t: \Pi TX^g \rightarrow \Pi TX^g \times \mathbb{R}^{1|1}$, $x \mapsto (x, t)$, for $t = 0, 1$, and denote by $\text{SP}: c_0^*W \rightarrow c_1^*W$ the parallel transport operator along that superinterval. There is a slight subtlety to notice here. Since the maps $c_0 = c \circ i_0$ and $c_1 = c \circ i_1$ are equal, c_0^*W and c_1^*W are the same vector bundle, but the correct way to identify them (for the purposes of computing the supertrace) is via the action of g . Indeed, let us form the pullback of principal bundles

$$\begin{array}{ccccc} \tilde{Q} & \xrightarrow{\quad} & Q & \xrightarrow[f]{} & X \\ \downarrow & & \downarrow & & \\ \Pi TX^g \times \mathbb{R}^{1|1} & \xrightarrow{c} & K & & \end{array}$$

(Note: A curved arrow labeled \tilde{f} also connects \tilde{Q} to X .)

where $\tilde{Q} = \Pi TX^g \times \mathbb{R}^{1|1} \times G$. Then the pullback c^*W can be identified with the restriction of the pullback of V to the identity section of \tilde{Q} ,

$$c^*W \cong (\tilde{f}^*V)/G \cong (\tilde{f}^*V)|_{\Pi TX^g \times \mathbb{R}^{1|1} \times \{e\}},$$

so we identify

$$c_0^*W_x = \tilde{f}^*V_{(x,0,e)} = \tilde{f}^*V_{(x,1,g)} \xrightarrow{g^{-1}} \tilde{f}^*V_{(x,1,e)} = c_1^*W_x.$$

We write, as before, $V|_{X^g} = V^1 \oplus \dots \oplus V^r$ as a direct sum of eigenspaces for eigenvalues $\lambda_1, \dots, \lambda_r$, with connection ∇_i on each component. Since $\tilde{f}|_{\Pi TX^g \times \mathbb{R}^{1|1} \times \{e\}}$ takes values in X^g , this induces a similar decomposition of c^*W into a sum of vector bundles \tilde{W}^i with connection. We are finally ready to invoke the calculations of Dumitrescu [7] recovering the usual (nonequivariant) Chen character in terms of parallel transport. Denoting by $\text{SP}_i: \tilde{W}^i|_{\Pi TX^g \times 0} \rightarrow \tilde{W}^i|_{\Pi TX^g \times 1}$ the super parallel transport for one unit of time on each \tilde{W}^i , the calculations in the mentioned reference show that $\text{SP}_i = \exp(-\nabla_i^2)$, so that

$$\text{ch}(\nabla_i) = \text{str}(\exp(-\nabla_i^2)) = \text{str}(\text{SP}_i).$$

The original parallel transport $\text{SP}: c_0^*W \rightarrow c_1^*W \cong c_0^*W$ can be expressed as the composition

$$c_0^*W = \bigoplus_i \tilde{W}^i|_{\Pi TX^g \times 0} \xrightarrow{\oplus_i \text{SP}_i} \bigoplus_i \tilde{W}^i|_{\Pi TX^g \times 1} = c_1^*W \xrightarrow{g} c_0^*W$$

and we conclude that

$$\text{str}(\text{SP}) = \sum_{1 \leq i \leq r} \text{str}(g \text{SP}_i) = \sum_{1 \leq i \leq r} \lambda_i \text{str}(\text{SP}_i) = \sum_{1 \leq i \leq r} \lambda_i \text{ch}(\nabla_i)$$

is a differential form representative of $\text{ch}_g(V)$. Next, we notice that the differential forms thus obtained on each ΠTX^g determine a G -invariant form on $\Pi T\hat{X} = \Pi_{g \in G} \Pi TX^g$. This form is precisely the function on $\Pi T\hat{X}$ associated to (8) via $\text{red}(Z_V)$, and it represents $\text{ch}_G(V)$ in de Rham cohomology. This finishes the proof of theorem 1.

APPENDIX A. GROUP ACTIONS ON STACKS

We briefly review the definitions of group action on a stack and quotient of a stack, following Romagny [13] and Ginot and Noohi [10], and then prove a useful lemma (proposition 8). Note that limits and colimits here are always taken in the sense of bicategories. These are often called 2-(co)limits, bi(co)limits or homotopy (co)limits.

A.1. Basic definitions. Let \mathfrak{X} be a groupoid fibration over a site \mathfrak{S} and G strict monoid object in the 2-category of fibrations over \mathfrak{S} . We denote by $m: G \times G \rightarrow G$ and $1: \text{pt} \rightarrow G$ the multiplication law and unit map of G . A (left) action of G on \mathfrak{X} is a map of groupoid fibrations $\mu: G \times \mathfrak{X} \rightarrow \mathfrak{X}$ together with (necessarily invertible) 2-morphisms α, \mathfrak{a} as in the diagram below.

$$\begin{array}{ccc} G \times G \times \mathfrak{X} & \xrightarrow{m \times \text{id}} & G \times \mathfrak{X} \\ \text{id} \times \mu \downarrow & \nearrow \alpha & \downarrow \mu \\ G \times \mathfrak{X} & \xrightarrow{\mu} & \mathfrak{X} \end{array} \quad \begin{array}{ccc} G \times \mathfrak{X} & \xrightarrow{\mu} & \mathfrak{X} \\ 1 \times \text{id} \uparrow & \nwarrow \mathfrak{a} & \uparrow \text{id} \\ \mathfrak{X} & & \end{array}$$

In formulas, given an object $x \in \mathfrak{X}_S$ and $g, h \in G_S$, and using a dot to denote the group action, we are given natural isomorphisms

$$\alpha_{g,h}^x: g \cdot (h \cdot x) \rightarrow (gh) \cdot x, \quad \mathfrak{a}^x: 1 \cdot x \rightarrow x.$$

This data is required to satisfy compatibility conditions that bear some resemblance to the axioms of a monoidal category. Firstly, a kind of pentagon identity relating the different ways in which the action of three group elements $g, h, k \in G_S$ can be associated:

$$\alpha_{g,hk}^x \circ g \cdot \alpha_{h,k}^x = \alpha_{gh,k}^x \circ \alpha_{g,h}^{g \cdot x}.$$

Second, a condition on the two ways of associating the action of the unit and another group element:

$$g \cdot \mathfrak{a}^x = \alpha_{g,1}^x \text{ and } \mathfrak{a}^{g \cdot x} = \alpha_{1,g}^x.$$

It seems appropriate to call α and \mathfrak{a} the associator and unitor for the action, in analogy to the terminology used in the theory of monoidal categories. We say the action is strict if α, \mathfrak{a} are both the identity.

Now, suppose we are given fibrations with G -action $(\mathfrak{X}, \mu, \alpha, \mathfrak{a})$ and $(\mathfrak{Y}, \nu, \beta, \mathfrak{b})$. A G -equivariant map between them is a morphism of fibrations $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ together with a 2-morphism

$$\begin{array}{ccc} G \times \mathfrak{X} & \xrightarrow{\mu} & \mathfrak{X} \\ \text{id} \times f \downarrow & \nearrow \rho & \downarrow f \\ G \times \mathfrak{Y} & \xrightarrow{\nu} & \mathfrak{Y} \end{array}$$

satisfying the following compatibility condition: for each $x \in \mathfrak{X}_S$ and $g, h \in G_S$, we have

$$f(\alpha_{g,h}^x) \circ \rho_g^{h \cdot x} \circ g \cdot \rho_h^x = \rho_{gh}^x \circ \beta_{g,h}^{f(x)} \text{ and } f(\mathfrak{a}^x) \circ \rho_1^x = \mathfrak{b}^{f(x)}.$$

We will call ρ the equivariance datum. Finally, a G -equivariant 2-morphism between morphisms $(f, \rho), (f', \rho')$ as above is given by a 2-morphism $\xi: f \rightarrow f'$ between the underlying fibered functors which is compatible with ρ, ρ' in the sense that

$$\rho_g'^x \circ g \cdot \xi^x = \xi^{g \cdot x} \circ \rho_g^x$$

for any $x \in \mathfrak{X}_S, g \in G_S$.

In terms of pasting diagrams, the conditions on ρ are expressed by the commutativity of the cube whose two halves are depicted below,

$$\begin{array}{ccc}
 G \times G \times \mathfrak{X} & \xrightarrow{m \times \text{id}} & G \times \mathfrak{X} \xrightarrow{\mu} \mathfrak{X} \\
 \downarrow & & \downarrow \\
 G \times G \times \mathfrak{Y} & \xrightarrow{\text{id} \times \nu} & G \times \mathfrak{Y} \xrightarrow{\nu} \mathfrak{Y} \\
 & \searrow m \times \text{id} & \nearrow \beta \\
 & G \times \mathfrak{Y} &
 \end{array}
 \quad
 \begin{array}{ccc}
 G \times G \times \mathfrak{X} & \xrightarrow{\text{id} \times \mu} & G \times \mathfrak{X} \\
 \downarrow & \nearrow m \times \text{id} & \downarrow \\
 G \times G \times \mathfrak{Y} & \xrightarrow{\text{id} \times \nu} & G \times \mathfrak{Y} \\
 & \nearrow \text{id} \times \rho & \downarrow \\
 & G \times \mathfrak{Y} &
 \end{array}$$

and commutativity of the prism

$$\begin{array}{ccccc}
 & & G \times \mathfrak{X} & & \\
 & \nearrow 1 \times \text{id} & \downarrow & \searrow \mu & \\
 \mathfrak{X} & & G \times \mathfrak{Y} & & \mathfrak{X} \\
 \downarrow & \nearrow \text{id} & \downarrow & \searrow \rho & \downarrow \\
 \mathfrak{Y} & & G \times \mathfrak{Y} & & \mathfrak{Y} \\
 & \nearrow 1 \times \text{id} & \downarrow & \searrow \nu & \\
 & & G \times \mathfrak{Y} & & \\
 & & \downarrow & & \\
 & & \mathfrak{Y} & &
 \end{array}$$

Here, all vertical maps are products of f and the identity of G . The condition on ξ is the commutativity of the following diagram.

$$\begin{array}{ccc}
 G \times \mathfrak{X} & \xrightarrow{\mu} & \mathfrak{X} \\
 \downarrow \text{id} \times f & \nearrow \rho' & \downarrow f \\
 G \times \mathfrak{Y} & \xrightarrow{\nu} & \mathfrak{Y}
 \end{array}$$

We are mostly interested in the case where G is a (representable) sheaf of groups, but we will also consider the group stack $\text{pt} // \mathbb{Z}$, assigning to any $S \in \mathfrak{S}$ the groupoid $\mathbb{Z} \rightrightarrows \text{pt}$. Note that a strict action of $\text{pt} // \mathbb{Z}$ on a stack \mathfrak{X} is precisely the data of an automorphism of $\text{id}_{\mathfrak{X}}$, i.e., a natural choice of automorphism for each object of \mathfrak{X} . For instance, the inertia stack $\Lambda \mathfrak{X}$ comes with a canonical $\text{pt} // \mathbb{Z}$ -action. We will also make use of a 2-categorical model for the circle group to be denoted $\mathbb{R} // \mathbb{Z}$. It is presented by the Lie 2-group $\mathbb{Z} \times \mathbb{R} \rightrightarrows \mathbb{R}$ (the transport groupoid of the \mathbb{Z} -action on \mathbb{R}) endowed with the multiplication map determined by the group structures on the spaces of objects and morphisms, and unit $0 \in \mathbb{R}$. At the Lie 2-group level, there are evident strict homomorphisms

$$\mathbb{T} \leftarrow \mathbb{R} // \mathbb{Z} \rightarrow \text{pt} // \mathbb{Z}.$$

The left map gives us an equivalence of group stacks, but in concrete situations it may be more convenient to consider one model or the other.

A.2. Quotient stacks of G -stacks. Let \mathfrak{X} be a stack endowed with a left action of a sheaf of groups G . Then we define a new stack $G \backslash \mathfrak{X}$ whose S -points are given by

a left G -torsor $P \rightarrow S$ together with a G -equivariant map $\psi: P \rightarrow \mathfrak{X}$; a morphism $(P', \psi') \rightarrow (P, \psi)$ covering $f: S' \rightarrow S$ is given by a diagram

$$\begin{array}{ccccc} & & \psi' & & \\ & \nearrow & \Downarrow \xi & \searrow & \\ P' & \xrightarrow{\Phi} & P & \xrightarrow{\psi} & \mathfrak{X} \\ \downarrow & & \downarrow & & \\ S' & \xrightarrow{f} & S & & \end{array}$$

where Φ is a map of G -torsors and ξ an equivariant 2-morphism.

There is a faithful functor $i: \mathfrak{X} \rightarrow G \backslash \backslash \mathfrak{X}$ sending $x: S \rightarrow \mathfrak{X}$ to the S -point of $G \backslash \backslash \mathfrak{X}$ consisting of the trivial G -torsor $G \times S \rightarrow S$ together with the G -equivariant map

$$\psi: G \times S \xrightarrow{\text{id} \times x} G \times \mathfrak{X} \xrightarrow{\mu} \mathfrak{X}.$$

This makes the diagram below 2-cartesian.

$$\begin{array}{ccc} G \times \mathfrak{X} & \xrightarrow{\mu} & \mathfrak{X} \\ \text{pr}_2 \downarrow & & \downarrow i \\ \mathfrak{X} & \xrightarrow{i} & G \backslash \backslash \mathfrak{X} \end{array}$$

Now, we can attempt to perform the construction of a transport groupoid $G \times \mathfrak{X} \rightrightarrows \mathfrak{X}$ internally in the 2-category of stacks. For this to work, we need to define internal categories with the appropriate degree of weakness (e.g., if the action is not strictly unital, the same must be allowed of our internal categories). In any case, it is clear that we get a “nerve”, that is, an augmented (weak) simplicial object

$$(9) \quad G \backslash \backslash \mathfrak{X} \xleftarrow{i} \mathfrak{X} \rightrightarrows G \times \mathfrak{X} \xleftarrow{\mu} G \times G \times \mathfrak{X} \xleftarrow{\mu} \dots$$

Since the various compositions $G^n \times \mathfrak{X} \rightarrow G \backslash \backslash \mathfrak{X}$ are not equal, just isomorphic (with a specified isomorphism), the augmentation depends, strictly speaking, on a choice. For definiteness, we take that to be the composition of i with the projection $\text{pr}_{n+1}: G^n \times \mathfrak{X} \rightarrow \mathfrak{X}$.

Proposition 7. *The above induces an equivalence of stacks*

$$G \backslash \backslash \mathfrak{X} \xleftarrow{j} \text{colim} \left(\mathfrak{X} \rightrightarrows G \times \mathfrak{X} \xleftarrow{\mu} G \times G \times \mathfrak{X} \xleftarrow{\mu} \dots \right).$$

The reader well versed on colimits of categories may be able to interpret the discussion in sections 3.2 and 4.2 of Ginot and Noohi’s paper [10] as a proof, even though it does not use the language of colimits. In any case, we will provide our own argument. Before getting there, we give some background on (homotopy) colimits in Cat . Given a diagram of small categories $F: D \rightarrow \text{Cat}$ indexed by a small 1-category (with no strictness requirements on F), we denote by $D \ltimes F$ the Grothendieck construction. It is the oplax colimit of F , meaning that for each $C \in \text{Cat}$, there is an equivalence between the category of functors $D \ltimes F \rightarrow C$ and the category of lax natural transformations $F \rightarrow \text{const}_C$ and modifications between them. The colimit of F is obtained by localizing $D \ltimes F$ at the class of opcartesian morphisms.

Spelling out the above, the colimit can be described in terms of generators and relations as follows. We write i, j , etc., for objects of D and A_i, A_j for their images via F ; also, we use the same notation both for a morphisms $f: i \rightarrow j$ in D and

its image $f: A_i \rightarrow A_j$. To build $A = \operatorname{colim}_D A_i$, we start with the disjoint union $\coprod_{i \in D} A_i$ and then freely adjoin inverse morphisms

$$f_x: x \rightarrow f(x), \quad f_x^{-1}: f(x) \rightarrow x$$

for each $f: i \rightarrow j$ in D and $x \in A_i$; finally, we impose a number of natural relations, most notably

$$\left(x \xrightarrow{\phi} y \xrightarrow{f_y} f(y) \right) = \left(x \xrightarrow{f_x} f(x) \xrightarrow{f(\phi)} f(y) \right),$$

where ϕ is a morphism in A_i , as well as its counterpart involving f_x^{-1} , f_y^{-1} . This process can be made precise using the free category generated by a directed graph and congruences. For more details, including the proof that this has the desired universal property, see Fiore [9, chapter 4].

Proof of proposition 7. Colimits of stacks are obtained by taking colimits objectwise in \mathfrak{S} and then stackifying. Thus, it suffices to show that, for each $S \in \mathfrak{S}$,

$$(G \backslash \mathfrak{X})_S \xleftarrow{j_S} \operatorname{colim} \left(\mathfrak{X}_S \leftarrow (G \times \mathfrak{X})_S \xleftarrow{\quad} (G \times G \times \mathfrak{X})_S \xleftarrow{\quad} \cdots \right)$$

gives an equivalence of the right-hand side with the full subgroupoid $(G \backslash \mathfrak{X})_S^{\operatorname{triv}}$ of the left-hand side involving only trivial G -torsors. To simplify the argument, we assume, without loss of generality, that the G_S -action on \mathfrak{X}_S is strict [13, proposition 1.5].

Consider the functor $l: (G \backslash \mathfrak{X})_S^{\operatorname{triv}} \rightarrow \operatorname{colim}_n (G^n \times \mathfrak{X})_S$ prescribed by the following conditions. First, on \mathfrak{X}_S , seen as a subgroupoid of both the domain (via $i: \mathfrak{X}_S \hookrightarrow (G \backslash \mathfrak{X})_S^{\operatorname{triv}}$) and codomain, l is just the identity. Second, to the morphism $x \rightarrow g \cdot x$ in $(G \backslash \mathfrak{X})_S^{\operatorname{triv}}$ determined by $g \in G_S$, l associates the morphism

$$\mu_g^x: x \xrightarrow{\operatorname{pr}_2^{-1}} (g, x) \xrightarrow{\mu} g \cdot x$$

in the colimit groupoid. To see that this is well defined and respects compositions, it suffices to check that the outer square of the following diagram in the colimit groupoid commutes, for any $g, h \in G_S$ and $\xi: g \cdot x \rightarrow y$ in \mathfrak{X}_S .

$$\begin{array}{ccccc} g \cdot x & \xrightarrow{\quad \xi \quad} & y & & \\ \downarrow \mu_h^{g \cdot x} & \swarrow \operatorname{pr}_2 & \searrow \operatorname{pr}_2 & & \downarrow \mu_h^y \\ & (h, g \cdot x) \xrightarrow{\operatorname{id} \times \xi} (h, y) & & & \\ & \swarrow \mu & \searrow \mu & & \\ hg \cdot x & \xrightarrow{\quad h \cdot \xi \quad} & h \cdot y & & \end{array}$$

This follows from the fact that each circuit traveling inside the square commutes.

Now, the composition $j_S \circ l$ is equal to the identity, and we claim that the reverse composition is isomorphic to the identity. In fact, $l \circ j_S(g_1, \dots, g_n, x) = x$, and we define a natural transformation $u: \operatorname{id} \rightarrow l \circ j_S$ by

$$u_{(g_1, \dots, g_n, x)} = \operatorname{pr}_{n+1}: (g_1, \dots, g_n, x) \rightarrow x.$$

Naturality with respect to those morphisms in the colimit groupoid which arise from morphisms in $(G^n \times \mathfrak{X})_S$ is obvious. A general morphism arising from the indexing

category Δ^{op} is as in the left vertical arrow of the diagram below,

$$\begin{array}{ccc}
 (g_1, \dots, g_n, x) & \xrightarrow{\text{pr}_{n+1}} & x \\
 \downarrow & \searrow \text{pr}_2 & \downarrow \mu_{g_J}^x \\
 & (g_J, x) & \\
 (g_{I_1}, \dots, g_{I_k}, g_J \cdot x) & \xrightarrow{\text{pr}_{k+1}} & g_J \cdot x
 \end{array}$$

where, $I_1, \dots, I_k, J \subset [n]$ are (possibly empty) disjoint and adjacent subsets whose union contains n , and $g_{\{i_1, \dots, i_j\}} = g_{i_1} \dots g_{i_j}$. Its image through $l \circ j_S$ is the right vertical arrow, and naturality of u , that is, the claim that the outer square commutes, follows from commutativity of the circuits involving (g_J, x) . This finishes the proof that j_S is an equivalence onto $(G \backslash \mathfrak{X})_S^{\text{triv}}$. \square

Now, given a stack \mathfrak{C} , applying $\text{Fun}_{\mathfrak{S}}(-, \mathfrak{C})$ to diagram (9) produces a (weak) cosimplicial groupoid. The following descent calculation for G -stacks is then a corollary of proposition 7.

Proposition 8. *For any stack \mathfrak{C} and G -stack \mathfrak{X} , diagram (9) induces an equivalence of groupoids*

$$\text{Fun}_{\mathfrak{S}}(G \backslash \mathfrak{X}, \mathfrak{C}) \cong \lim \left(\text{Fun}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{C}) \rightrightarrows \text{Fun}_{\mathfrak{S}}(G \times \mathfrak{X}, \mathfrak{C}) \rightrightarrows \dots \right).$$

Again, a concrete description of 2-limits in the 2-category of small categories can be found in Fiore [9, chapter 5]. For the convenience of the reader, we give a quick summary here. We fix the same notations as in the discussion of colimits above; in particular, we have a diagram $F: D \rightarrow \text{Cat}$. Then (a model for) the limit of F is the category whose objects are (pseudo) natural transformations $\Delta_{\text{pt}} \rightarrow F$ with domain the constant functor with value the discrete category with one object, and whose morphisms are modifications between them. In concrete terms, an object consists of a collection of objects $a_i \in A_i$ for each $i \in D$ together isomorphisms $\tau_f: f(a_i) \rightarrow a_j$ for each morphism $f: i \rightarrow j$ in D ; these data are required to satisfy certain coherence conditions. A morphism $(a'_i, \tau'_f) \rightarrow (a_i, \tau_f)$ consists of a collection of morphisms $a'_i \rightarrow a_i$ in A_i for each $i \in D$, subject to appropriate conditions.

APPENDIX B. LOW-DIMENSIONAL EUCLIDEAN SUPERGEOMETRY

In the category SM of supermanifolds, $\mathbb{R}^{1|1}$ has a (noncommutative) group structure given by

$$\mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \rightarrow \mathbb{R}^{1|1}, \quad ((t, \theta), (t', \theta')) \mapsto (t + t' + \theta\theta', \theta + \theta').$$

The Lie algebra of left-invariant vector fields is free on one odd generator $D = \partial_{\theta} - \theta\partial_t$, and actions of $\mathbb{R}^{1|1}$ correspond (bijectively, modulo noncompactness issues) to odd vector fields. Similarly, $\mathbb{R}^{0|1}$ has Lie algebra spanned by an odd element ∂_{θ} squaring to 0, and its actions correspond bijectively to homological vector fields.

The definition of Euclidean structures on supermanifolds follows the philosophy of Felix Klein's Erlangen program. One starts by fixing a model space and a subgroup of diffeomorphisms, called the isometry group; a Euclidean structure is then a maximal atlas whose transition maps are isometries. This idea is explained in detail

in Stolz and Teichner [17, sections 2.5 and 4.2]. In (real) dimensions 0|1 and 1|1, the model spaces are $\mathbb{R}^{0|1}$ respectively $\mathbb{R}^{1|1}$ with isometry groups

$$\mathrm{Isom}(\mathbb{R}^{0|1}) = \mathbb{R}^{0|1} \rtimes \mathbb{Z}/2, \quad \mathrm{Isom}(\mathbb{R}^{1|1}) = \mathbb{R}^{1|1} \rtimes \mathbb{Z}/2.$$

In both cases, $\mathbb{Z}/2$ acts by negating the odd coordinate and $\mathbb{R}^{d|1}$ acts by *left* multiplication (this choice influences our sign conventions, and dictates whether to work with left of right group actions at various places).

The differential form $-\theta d\theta$ on $\mathbb{R}^{0|1}$ is invariant under isometries, and therefore defines a canonical fiberwise 1-form ζ on any family of Euclidean 0|1-manifolds $\Sigma \rightarrow S$. Similarly, the form $dt - \theta d\theta$ on $\mathbb{R}^{1|1}$ determines a canonical fiberwise 1-form ω on any family of Euclidean 1|1-manifolds. It is possible to characterize low-dimensional Euclidean structures in terms of these data, which can be seen as a sort of odd analogues of symplectic and contact structures.

Theorem 9. *Let $\Sigma \rightarrow S$ be an S -family of Euclidean 0|1-manifolds and $P \rightarrow \Sigma$ a principal \mathbb{T} -bundle. Then a fiberwise (in S) connection form ω on P whose curvature agrees with the tautological 2-form $d\zeta$ on Σ canonically determines a Euclidean structure on P . Isometries of P correspond to connection-preserving bundle maps covering an isometry of Σ .*

See for instance [15, section 3.3] for a proof. To see why the data of ω is essential here, notice that the short exact sequence of super Lie groups

$$1 \rightarrow \mathbb{R} \rightarrow \mathbb{R}^{1|1} \rightarrow \mathbb{R}^{0|1} \rightarrow 1$$

is not split. As a consequence, the cartesian product of Euclidean manifolds of dimensions 1 and 0|1 is not endowed with a canonical Euclidean structure. This makes dimensional reduction in this setting a bit subtle, since “crossing with S^1 ” is not a well-defined operation in the Euclidean category.

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